MATH5012 Exercise 2

Here μ is a Radon measure on $\mathbb{R}^n$. Many problems are taken from [R1].

(1) Use maximal function to give another proof of Lebesgue differentiation theorem. Setting

$$(T_r f)(x) = \frac{1}{\mu(\overline{B}_r(x))} \int_{\overline{B}_r(x)} |f - f(x)| d\mu \ ,$$

and

$$(Tf)(x) = \limsup_{r \to 0} (T_r f)(x) \ .$$

Show that $T_r f = 0$ μ-a.e.. Suggestion: For $\varepsilon > 0$, pick continuous g such that $\|f - g\|_{L^1} < \varepsilon$ and establish $Tf(x) \le Mh(x) + |h|(x)$ where $h = f - g$. Then use 7(a) in Ex 1.

(2) Let E be μ-measurable. Show that μ-a.e. $x \in \mathbb{R}^n \setminus E$ has density 0 in E.

(3) Let F be closed in $\mathbb{R}$ and $\delta(x)$ the distance from x to F,

$$\delta(x) = \inf \{|x - y| : y \in F\} \, .$$

Show that

$$\frac{\delta(x+y)}{|y|} \to 0 \quad \text{a.e. } x \in F \text{ as } y \to 0.$$

Hint: May take x a point of density 1.

(4) For $\delta > 0$, let $I(\delta) = (-\delta, \delta)$. Given α and β, $0 \le \alpha < \beta \le 1$, construct a measurable set E so that the upper and lower limits of $\mathcal{L}^1(E \cap I(\delta))/2\delta$ are equal to α and β respectively as $\delta \to 0$.

(5) If $A \subset \mathbb{R}^1$ and $B \subset \mathbb{R}^1$, define $A + B = \{a + b : a \in A, b \in B\}$. Suppose $m(a) > 0$, $m(b) > 0$. Prove that $A + B$ contains a segment, by completing the outline given in [R1].

(6) A point $x \in \mathbb{R}^n$ is called an atom for a measure λ if $\lambda(\{x\}) > 0$. Establish the decomposition

$$\mu = f\mathcal{L}^n + \mu_{cs} + \sum_k a_k \delta_{x_k}, \quad a_k > 0,$$

where $f \in L^1(\mathcal{L}^n)$ and μ_{cs} has no atoms.

(7) Let $\{x_n\}$ be an infinite sequence of distinct numbers in $[0, 1]$. Can you find an increasing function in $[0, 1]$ whose discontinuity set is precisely $\{x_n\}$?

(8) (a) Consider the real line. Show that x is not an atom for μ if and only if its distribution function is continuous at x.

(b) Use (a) to construct a singular measure, that is, perpendicular to $\mathcal{L}^1$, without atoms. Suggestion: Consider the Cantor-Lebesgue function.

(9) Let μ be a singular measure with respect to $\mathcal{L}^1$ and f its distribution function. Show that for μ-a.e. x, either f'_+ or f'_- becomes ∞.

(10) Construct a continuous monotonic function f or $\mathbb{R}^1$ so that f is not constant on any segment although $f'(x) = 0$ a.e.